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SOLUTION OF SINGLY-CONSTRAINED CONCAVE ALLOCATION PROBLEMS,(U)
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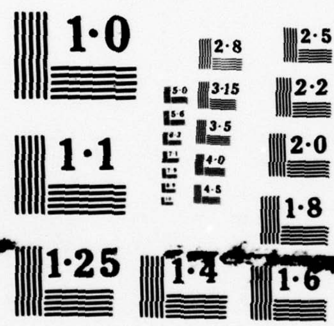
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RESEARCH PAPER P-619

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James T. McGill

January 1970

INSTITUTE FOR DEFENSE ANALYSES
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FOREWORD

The motivation for this paper emanated from a problem formulated by W. J. Schultis[1] and subsequently addressed by C. L. Dym and R. E. Schwartz.[2] The work was supported by the Institute for Defense Analyses.

The author is indebted to R. E. Schwartz for a strengthening of the original Theorem 11 and for suggesting the result contained in Theorem 12.

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1. W. J. Schultis, A Manual Model for Strategic Conflict Analysis, IDA Research Paper P-493 (July 1969).

2. C. L. Dym and R. E. Schwartz, Optimum Pre-Attack Targeting, IDA Research Paper P-546 (October 1969).

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ABSTRACT

The problem of finding an optimal integer allocation $x = (x_1, x_2, \ldots, x_n)$ for

$$\text{maximizing } \sum_{i=1}^{n} f_i(x_i)$$

subject to

$$\sum_{i=1}^{n} g_i(x_i) \leq m$$

$$x_i \geq 0$$

is considered. For $g_i(x_i) \equiv x_i$, a class of objective functions is characterized for which a simple rounding procedure leads from an optimal continuous solution to an optimal integer solution. The rounding procedure developed at IDA by C. L. Dym and R. E. Schwartz for a targeting allocation problem is shown to be a special case of a more general procedure.

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The problem of finding an optimal integer allocation $x = (x_1, x_2, \ldots, x_n)$ for

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I

INTRODUCTION

The optimization technique known as mathematical programming seeks the solution of the following problem: find x to

$$\text{maximize } f(x)$$

subject to

$$g_i(x) \leq b_i \ , \quad i = 1, 2, \ldots, n,$$

$$x \in X \ .$$

Research efforts in the area of mathematical programming can be dichotomized by the generality of the problem considered. Much of the expended research resources are devoted to developing computational algorithms for, or characterizations of, the solutions to very general classes of such problems. The generality of the results thus obtained is, without quarrel, useful, elegant, and pedagogically pleasing. However, the broadness of the problems treated in such generality must, of necessity, also engender quite general assumptions about their structure. A case in point is the class of convex programming problems.[1] Many elegant results exist for these problems, and several algorithms are particularly efficient for the computation of the optimal solution.

Another line of research considers the analysis of mathematical programs which are specially structured. The successful analysis

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1. A mathematical program for maximizing is a convex program if (1) $f(x)$ is concave, (2) $g_i(x)$ is convex, $i=1,2,\ldots,n$, and (3) X is a convex set.

of such problems exploits their structure so as to improve, either theoretically or computationally, upon the known results for more general problems. It is this realm into which the content of this paper rightfully falls.

Before stating the particular problem of concern herein, a further discursive remark is in order. As is well known, the requirement that the solution to a mathematical program be integer typically introduces severe analytical difficulties. There are two general approaches to solving integer problems. The algorithms presently described in the literature reflect both approaches. First, the continuous problem may be solved, and then by some perturbation of the continuous solution an optimal integer solution is calculated. Rounding the continuous solution in some manner is an example of such a perturbation. On the other hand, the integer problem may be directly approached without regard for the underlying continuous problem. Both approaches are considered for the problem treated in this paper.

An important class of practical problems requires the determination of an allocation of a resource to a set of independent alternative uses of that resource. The decisionmaker wants to maximize the return accruing from the allocation of the resource. Suppose the number of alternative uses available is n . Let $x = (x_1, x_2, \ldots, x_n)$ be an allocation; that is, x_i is the amount of the resource assigned to alternative i . Suppose that $f_i(\cdot)$ is the return function for alternative i and $g_i(\cdot)$ is the resource utilization function for alternative i . Further, let m be the amount of the resource available to allocate. The singly-constrained allocation problem is to find x which solves:

$$\text{maximize} \sum_{i=1}^{n} f_i(x_i) \qquad (1)$$

subject to

$$\sum_{i=1}^{n} g_i(x_i) \le m \quad , \qquad (2)$$

$$x_i \geq 0 \quad , \qquad i = 1, 2, \ldots, n \quad , \tag{3}$$

$$x_i \text{ a nonnegative integer} \quad , \qquad i = 1,2,\ldots, n \quad . \tag{4}$$

The problem given by Eqs. 1, 2, and 3 will be called the "continuous problem," while the addition of the constraints given by Eq. 4 yields the "integer problem." The solution of either problem will be called "optimal." In Section II the continuous problem is treated in detail. By making certain assumptions about the functions $f_i(\cdot)$ and $g_i(\cdot)$, the optimal solution is characterized, and an algorithm is given for its computation. With added restrictions a similar treatment is given to the integer problem in Section III. Section IV addresses the interrelationship between the continuous and integer problems. Contained in that section are results which characterize a subclass of problems for which a naive rounding procedure yields an optimal integer solution from the optimal continuous solution. The paper concludes with some examples which are presented in Section V.

II

CONTINUOUS PROBLEM

The following assumptions are retained throughout this section:

A1: $f_i(\cdot)$ and $g_i(\cdot)$ are nonnegative, real-valued, and twice differentiable functions defined on the nonnegative real line;

A2: $f_i(\cdot)$ and $g_i(\cdot)$ are strictly increasing functions; and

A3: $f_i(\cdot)$ is a strictly concave function, and $g_i(\cdot)$ is a strictly convex function.

These assumptions are restrictive, but are satisfied by certain real-world problems, some examples of which are given in Section V. The assumption of the nonnegativity of $f_i(\cdot)$ and the strictness requirements in A2 and A3 are not crucial to the development in this section. These assumptions are made here for expository ease in the derivation of the results. Without loss of generality it can be assumed that $g_i(0) = 0$ for $i = 1, 2, \ldots, n$. To avoid trivialities, also assume that $m > 0$.

The above stated assumptions guarantee that a solution satisfying the Kuhn-Tucker optimality conditions (KTC) is optimal.[1] The KTC are:[2]

$$f_i'(x_i) - \lambda g_i'(x_i) \leq 0 \quad , \tag{5}$$

$$x_i[f_i'(x_i) - \lambda g_i'(x_i)] = 0 \quad , \tag{6}$$

1. These conditions can be found in most mathematical programming texts. The constraint qualification is not necessary here in that an interior point is guaranteed.

2. The notation $f'(x)$ denotes the first derivative of $f(\cdot)$ evaluated at x .

$$\lambda[m - \sum_{i=1}^{n} g_i(x_i)] = 0 \quad , \tag{7}$$

$$\lambda \geq 0 \quad , \tag{8}$$

in addition to Eqs. 2 and 3.

The remainder of this section addresses the problem of finding values of λ and x which satisfy the KTC. These values are first characterized in three lemmas which culminate in a reformulation of the problem. An algorithm based on the new problem is then given, providing a method of constructing the optimal solution.

Lemma 1: If (λ, x) satisfies the KTC, then $\lambda > 0$.

Proof:

Suppose $\lambda = 0$. Then from Eq. 5, $f_i'(x_i) \leq 0$ for all i which contradicts assumption A2. Thus, $\lambda > 0$. Q.E.D.

Define the function $h_i(\cdot)$ as follows:

$$h_i(x_i) = \frac{f_i'(x_i)}{g_i'(x_i)} \quad ,$$

for $x_i \geq 0$. Lemma 2 gives some useful properties of this function.

Lemma 2: The function $h_i(\cdot)$ is positive and strictly decreasing on $[0,\infty]$. Hence, its inverse function $h_i^{-1}(\cdot)$ is uniquely defined and is continuous on $[h_i(\infty), h_i(0)]$.

Proof:

By assumption A2, $f_i'(\cdot)$ and $g_i'(\cdot)$ are positive, and so $h_i(\cdot)$ is positive. Now for $x_i \geq 0$,

$$\frac{d}{dx_i} h_i(x_i) = \frac{g_i'(x_i)f_i''(x_i) - f_i'(x_i)g_i''(x_i)}{[g_i'(x_i)]^2} \quad .$$

By assumptions A2 and A3, the numerator is negative. Thus, $h_i(\cdot)$ is strictly decreasing from $h_i(\infty)$ to $h_i(0)$.

Furthermore from assumption A1, $h_i(\cdot)$ is a continuous function. Thus $h_i^{-1}(y)$ exists, is unique, and is continuous on $[h_i(\infty), h_i(0)]$. Q.E.D.

Letting $x_i(\lambda)$ denote a solution to Eqs. 5 and 6 for a fixed value of λ , Lemma 3 characterizes this solution as a function of λ. Theorem 4 follows, providing the main result of this section.

Lemma 3: Suppose that $\lambda \geq h_i(\infty)$. Then the unique value of x_i, denoted $x_i(\lambda)$, which solves Eqs. 5 and 6 is given by

$$x_i(\lambda) = \begin{cases} 0 \quad , & \text{if } \lambda > h_i(0) \quad , \\ h_i^{-1}(\lambda), & \text{if } h_i(\infty) \leq \lambda \leq h_i(0) \quad . \end{cases} \tag{9}$$

Proof:

Note first that $f_i'(\infty) - \lambda g_i'(\infty) \leq 0$, since $\lambda \geq h_i(\infty)$. If $\lambda > h_i(0)$, then letting $x_i(\lambda) = 0$ satisfies both Eqs. 5 and 6. Furthermore if in this case $x_i(\lambda) > 0$, then Eq. 6 is not satisfied. Hence, the unique value of $x_i(\lambda)$ for $\lambda > h_i(0)$ is $x_i(\lambda) = 0$.

If $h_i(\infty) \leq \lambda \leq h_i(0)$, then by letting $x_i(\lambda) = h_i^{-1}(\lambda)$, both Eqs. 5 and 6 are again satisfied. Furthermore by Lemma 2, the solution is unique. Q.E.D.

Theorem 4: Let λ^* be the solution to the following mathematical program:

$$\text{minimize } \lambda$$

subject to

$$\sum_{i=1}^{n} g_i[x_i(\lambda)] \leq m \quad , \tag{10}$$

where $x_i(\lambda)$ is given by Eq. 9. There exists a unique λ^* and, furthermore, $x(\lambda^*) = [x_1(\lambda^*), x_2(\lambda^*),\ldots, x_n(\lambda^*)]$ solves the continuous problem given by Eqs. 1, 2, and 3 and satisfies Eq. 10 with equality.

Proof:

If $\lambda \geq \max_i \{h_i(0)\}$, then Eq. 10 is satisfied, since by Lemma 3, $x_i(\lambda) = 0$ for $i = 1,2,\ldots, n$. Thus, there is a feasible solution to the mathematical program. From Lemmas 2 and 3, $x_i(\lambda)$ is seen to be a strictly decreasing function of λ on $[h_i(\infty), h_i(0)]$. Thus, $g_i[x_i(\lambda)]$ is also strictly decreasing in λ on the same interval. For $\lambda = \min_i \{h_i(\infty)\}$,

$$\sum_{i=1}^{n} g_i(x_i(\lambda)) = \infty > m \quad .$$

Thus, $\sum_{i=1}^{n} g_i[x_i(\lambda)]$ is strictly decreasing from ∞ to 0 as λ increases from $\min_i \{h_i(\infty)\}$ to $\max_i \{h_i(0)\}$. Hence, there exists a unique λ^* such that Eq. 10 is satisfied with equality. But, then, $[\lambda^*, x(\lambda^*)]$ satisfies the KTC, and so $x(\lambda^*)$ solves the continuous problem. Q.E.D.

Theorem 4 provides the basis for algorithms to solve the continuous problem. Any systematic scheme for varying λ until Eq. 10 is satisfied with equality will suffice. In some cases an analytical derivation of that λ^* may be accomplished. An example is discussed in Section V.

III

INTEGER PROBLEM

The integer problem given by Eqs. 1,2,3, and 4 is treated in this section. The following assumptions are retained throughout:

B1: $f_i(\cdot)$ and $g_i(\cdot)$ are nonnegative functions defined on $I = \{0,1,2,...\}$. Furthermore, $g_i(\cdot)$ is integer valued;

B2: $f_i(\cdot)$ and $g_i(\cdot)$ are strictly increasing functions; and

B3: $f_i(\cdot)$ is a strictly concave function, and $g_i(\cdot)$ is a strictly convex function.

For completeness the well-known result characterizing the maximum of a concave function over a set of integers is given in Lemma 5. Figure 1 illustrates the proof.

Lemma 5: A concave function h(x) defined on I is maximized at x*, where[1]

$$x^* = \begin{cases} 0, & \text{if } \Delta h(x) \leq 0 \text{ for } x = 0,1,2,\ldots \ , \\ \bar{x}, & \text{if } \Delta h(\bar{x}-1) > 0 \geq \Delta h(\bar{x}) \text{ for some } \bar{x} \geq 1 \ , \\ \infty, & \text{if } \Delta h(x) \geq 0 \text{ for } x = 0,1,2,\ldots \ . \end{cases}$$

Proof:

If $\Delta h(x) < 0$ for all nonnegative integers, then

$$h(0) \geq h(1) \geq h(2) \ldots \ ,$$

and so $h(\cdot)$ is maximized at 0 . If $\Delta h(x) \geq 0$ for all nonnegative integers, then

$$h(x+1) \geq h(x)$$

1. The notation $\Delta h(x)$ denotes the first difference of $h(\cdot)$ evaluated at x; that is $\Delta h(x) = h(x+1) - h(x)$. Also $\Delta^2 h(x) = \Delta h(x+1) - \Delta h(x)$.

for all x . Hence $h(\cdot)$ is maximized at ∞ .

Since $h(x)$ is concave, $\Delta h(x)$ is a nonincreasing function; that is $\Delta h(x) \geq \Delta h(x+1)$ for $x = 0,1,2...$. Suppose that it is not true that $\Delta h(x) \leq 0$ for all x . Then there is an x such that $\Delta h(x) > 0$. If it is also not true that $\Delta h(x) \geq 0$ for all x , then there is an $x \geq 1$ such that

$$\Delta h(\overline{x}-1) > 0 \geq \Delta h(\overline{x}) \quad .$$

It remains to show that $\overline{x}$ maximizes $h(\cdot)$ in this case. Suppose that y maximizes $h(\cdot)$. For specificity suppose that $y > \overline{x}$. Now

$$\begin{aligned} h(\overline{x}) - h(y) &= \sum_{k=0}^{y-\overline{x}-1} [- \Delta h(\overline{x}+k)] \\ &\geq 0 \quad , \end{aligned}$$

since $0 \geq \Delta h(\overline{x}) \geq \Delta h(\overline{x} + 1) \geq \ldots$. Thus, $\overline{x}$ does maximize $h(\cdot)$. An analogous argument suffices if $y < \overline{x}$. Q.E.D.

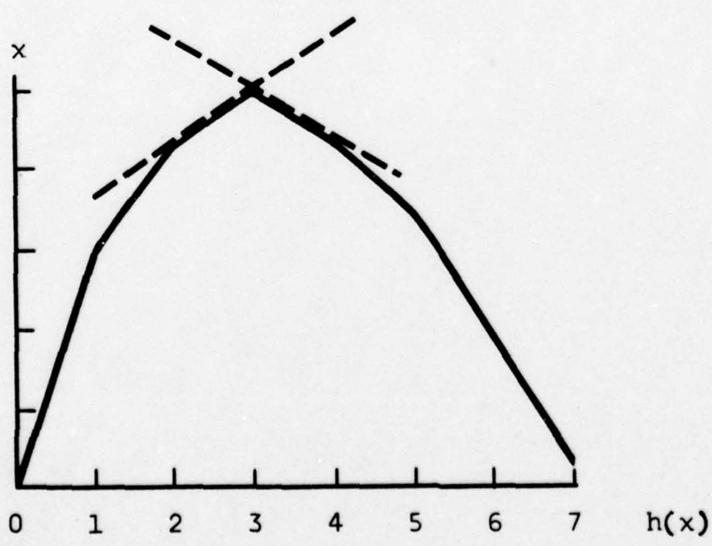

Figure 1. CONCAVE INTEGER-VALUED FUNCTION

Consider now the solution of the following (unconstrained Lagrangian) problem:

$$\max_{x} \sum_{i=1}^{n} [f_i(x_i) - \lambda g_i(x_i)] \tag{11}$$

$$x_i \in I \ , \ i=1,2,\ldots, n \ , \tag{12}$$

where $\lambda \geq 0$. For fixed λ, let $x(\lambda) = (x_1(\lambda), x_2(\lambda), \ldots x_n(\lambda))$ denote the solution. Lemma 6 is an analog to Lemma 3.

Lemma 6: The solution of Eqs. 11 and 12 for $\lambda \geq 0$ is $x(\lambda)$, where

$$x_i(\lambda) = \begin{cases} 0 \ , \text{ if } \lambda \geq \Delta f_i(0)/\Delta g_i(0) \\ \bar{x}_i, \text{ if } \Delta f_i(\bar{x}-1)/\Delta g_i(\bar{x}-1) > \lambda \geq \Delta f_i(\bar{x}_i)/\Delta g_i(\bar{x}_i) \\ \qquad \text{for some integer } \bar{x}_i \geq 1 \\ \infty \ , \text{ if } \Delta f_i(x)\Delta g_i(x) \geq \lambda \text{ all } x \ , \end{cases} \tag{13}$$

for $i=1,2,\ldots, n$.

Proof:

The function given by Eq. 11 is maximized if and only if $f_i(x_i) - \lambda g_i(x_i)$ is maximized for each i. By assumption B3 this function is concave. The result now follows from Lemma 5. Q.E.D.

Lemma 7: The function $x_i(\lambda)$ is a nonincreasing function of λ.

Proof:

Suppose that $\lambda_1 > \lambda_2 \geq 0$. Let $h_i(\cdot) = \Delta f_i(\cdot)/\Delta g_i(\cdot)$.

If $h_i(x) \geq \lambda_1$ for all x, then $h_i(x) \geq \lambda_2$ for all x. Thus, $x_i(\lambda_1) = \infty$ implies that $x_i(\lambda_2) = \infty$.

If $\lambda_1 \geq \Delta h_i(0)$, then $x_i(\lambda_1) = 0$ and so in this case $x_i(\lambda_2) \geq x_i(\lambda_1)$.

Finally, suppose that $h_i(\overline{x}_i-1) > \lambda_1 \geq h_i(\overline{x})$ for some $\overline{x}_i \geq 1$. Then $h_i(\overline{x}_i-1) > \lambda_2$ and so $x_i(\lambda_2) \geq \overline{x}_i = x_i(\lambda_1)$, which completes the proof. Q.E.D.

Theorem 8: Suppose that $g_i(0) = 0$. There exists a $\lambda^* \geq 0$ solving

$$\text{minimize } \lambda$$

subject to

$$\sum_{i=1}^{n} g_i[x_i(\lambda)] \leq m \quad , \tag{14}$$

where $x_i(\lambda)$ is given by Eq. 13. If Eq. 14 is satisfied with equality, then $x(\lambda)$ is the solution to Eqs. 1, 2, 3, and 4.

Proof:

By an analogous argument to that given in Theorem 4, there does exist a λ^* solving the constrained minimization problem.

The proof is completed by appeal to H. Everett's theorem[2] regarding the solution of constrained optimization problems. Q.E.D.

In comparing Theorems 4 and 8 it is to be noted that for the continuous case it is not required that a value of λ exist such that the analog of Eq. 14 holds with equality in order to obtain the optimal continuous solution. Assumption A2 implies that equality will indeed hold. That equality is required in Eq. 14 is not vacuous for the integer case. An example will illustrate that there are cases for which the optimal solution to Eqs. 1, 2, 3, and 4 cannot be found by the method inherent in Theorem 8. In

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2. H. Everett, "Generalized Lagrange Multiplier Method for Solving Problems of Optimal Allocation of Resources," Operations Research, II (1963), pp.399-417.

the *terminology* of Everett[3], such an example illustrates the existence of gaps. It is interesting to note that these gaps can occur even in convex programming problems if integer solutions are required.

Suppose that n = 2. The functional values of $f_i(\cdot)$ and $g_i(\cdot)$ are given in Table 1. These functions satisfy assumptions B1, B2, and B3. Table 2 gives the values of $x_1(\lambda)$ and $x_2(\lambda)$ for selected

Table 1

FUNCTION VALUES FOR AN EXAMPLE

x	$f_1(x)$	$g_1(x)$	$\Delta f_1(x)/\Delta g_1(x)$	$f_2(x)$	$g_2(x)$	$\Delta f_2(x)/\Delta g_2(x)$
0	0	0	5	0	0	10
1	5	1	2	20	2	6
2	9	3		38	5	

values of λ. It is clear that the minimization of λ *subject* to Eq. 14 does not yield the optimal integer solution for m = 1. In the special case given by Corollary 9, however, the minimization of λ subject to Eq. 14 will yield an optimal solution to Eqs. 1, 2, 3, and 4.

Table 2

INTEGER SOLUTION FOR AN EXAMPLE

	$x_1(\lambda)$	$x_2(\lambda)$	$g_1[x_1(\lambda)] + g_2[x_2(\lambda)]$
$\lambda \geq 10$	0	0	0
$10 > \lambda \geq 6$	0	1	2
$6 > \lambda \geq 5$	0	2	5

3. Ibid.

Corollary 9: Suppose that $g_i(x_i) = x_i$ for $i = 1,2,\ldots, n$. Then if λ^* is the smallest value of λ subject to Eq. 14, $x(\lambda^*)$ is the optimal solution for the integer allocation problem.

Proof:

From Lemma 7, $x_i(\lambda)$ increases as λ decreases. Thus, $\sum_{i=1}^{n} x_i(\lambda)$ also increases as λ is decreased. Noting from the proof of Lemmas 5 and 6 that if $\Delta f_i(\overline{x}_i)/\Delta g_i(\overline{x}_i) = \lambda$, then $x_i(\lambda)$ and $x_i(\lambda) + 1$ both maximize $f_i(x_i) - \lambda g_i(x_i)$, it follows that by decreasing λ to some λ^*, Eq. 14 can be satisfied (perhaps nonuniquely) with equality. By Theorem 8, then, the proof is complete. Q.E.D.

IV

RELATIONSHIP BETWEEN CONTINUOUS AND INTEGER SOLUTIONS

In this section results are given which relate the continuous solution to the integer solution. These results have computational importance in that they provide the means for applying simple rounding techniques to obtain the optimal integer solution from the optimal continuous solution.

The problem treated here is that of Section III where $g_i(x_i) = x_i$ for $i = 1, 2, \ldots, n$. The main result is a characterization of a subclass of allocation problems with concave return functions for which the optimal continuous solution can be rounded up or down by at most one to obtain the optimal integer solution. A rounding rule is also derived in order to specify which of the variables are to be rounded up and which are to rounded down.

<u>Theorem 10</u>: Let $\overline{x} = (\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n)$ be the optimal continuous solution to Eqs. 1, 2, and 3, and let $x^* = (x_1^*, x_2^*, \ldots, x_n^*)$ be the optimal integer solution to Eqs. 1, 2, 3, and 4. If[1]

$$x_i^* = [\overline{x}_i] \text{ or } [\overline{x}_i] + 1, \text{ for } i = 1,2,\ldots, n \quad , \tag{15}$$

then upon reordering the indices so that

$$\Delta f_i([\overline{x}_i] \geq \Delta f_{i+1}([\overline{x}_{i+1}]) \quad , \tag{16}$$

for $i = 1,2,\ldots, n - 1$, the optimal integer solution is

1. For any real number x, let $[x]$ be the largest integer less than or equal to x. A number is said to be "rounded" to $\overline{x}$ if the value of $\overline{x}$ is either $[x]$ or $[x] + 1$.

$$x_i^* = \begin{cases} [\overline{x}_i] + 1 \ , & \text{for } i = 1, 2, \ldots, k \ , \\ [\overline{x}_i] \ , & \text{for } i = k+1, k+2, \ldots, n \ , \end{cases} \tag{17}$$

where $k = m - \sum_{i=1}^{n} [\overline{x}_i]$.

<u>Proof</u>:

By the proof of Theorem 4 and by assumption, $\sum_{i=1}^{n} \overline{x}_i = m$. Thus,

$$\sum_{i=1}^{n} [\overline{x}_i] \le m \le n + \sum_{i=1}^{n} [\overline{x}_i] \quad .$$

Furthermore, $k = m - \sum_{i=1}^{n} [\overline{x}]$ is an integer and $0 \le k \le n$. By hypothesis (Eq. 15) the optimal integer solution can be obtained by rounding up and/or down at most one from the optimal continuous solution. By Corollary 9 the optimal integer solution satisfies $\sum_{i=1}^{n} x_i^* = m$. Thus, exactly k of the continuous variables must be rounded up and $n-k$ rounded down. To complete the proof it is shown that those variables which are rounded up are given by the k variables for which the difference of $f_i([\overline{x}_i])$ is largest. Suppose that there is an optimal allocation $y = (y_1, y_2, \ldots, y_n)$ such that $y_k = [\overline{x}_k] + 1$ and $y_j = [\overline{x}_j]$, where

$$\Delta f_j([\overline{x}_j]) > \Delta f_k([\overline{x}_k]) \quad .$$

Compare this solution with the one where $y^* = y$ except that $y_k^* = [\overline{x}_k]$ and $y_j^* = [\overline{x}_j] + 1$. The difference in the maximands for y and y^* is

$$f_j([\overline{x}_j] + 1) + f_k([\overline{x}_k]) - f_j([\overline{x}_j]) - f_k([\overline{x}_k] + 1)$$

$$= \Delta f_j([\overline{x}_j]) - \Delta f_k([\overline{x}_k])$$

$$> 0 \quad .$$

Thus, the allocation y* has a higher return than the allocation y, which contradicts the optimality of y. Q.E.D.

In order to make computational use of Theorem 10, it is necessary to know that Eq. 15 holds. A natural question to ask is when can one be assured that a rounding procedure is indeed optimal. Theorem 11 below gives a sufficient condition for Eq. 15 to hold. The condition is a strong one in the sense that the set of objective functions satisfying it is relatively small. However, for some applications of interest the condition can be shown to hold. Examples of these applications are given in detail in Section V. Theorem 12 provides an equivalent condition to that of Theorem 11. These conditions in effect demand that the return functions for each of the n alternatives be two dimensional translates of each other.

<u>Theorem 11</u>: For $x_i(\lambda)$ given by Eq. 9, suppose that

$$f_i'(x_i(\lambda) + \delta) = f_j'(x_j(\lambda) + \delta)$$

for all $\delta \geq -\min(x_i(\lambda), x_j(\lambda))$ when $x_i(\lambda) > 0$ and $x_j(\lambda) > 0$. Let $\overline{x}(\overline{\lambda})$ be the optimal (continuous) solution to Eqs. 1, 2, and 3, and let x* be the optimal (integer) solution to Eqs. 1, 2, 3, and 4. Reorder the indices so that the fractional parts of the continuous solution satisfy $r_1 \geq r_2 \geq \ldots \geq r_n$, where

$$[\overline{x}_i(\overline{\lambda})] + r_i = \overline{x}_i(\lambda) \quad .$$

Then, there is a k , $1 \leq k \leq n$, such that

$$x_i^* = \begin{cases} [\bar{x}_i(\bar{\lambda})] + 1 \ , & \text{for } i=1,2,\ldots,k \\ [\bar{x}_i(\bar{\lambda})] \ , & \text{for } i=k+1, k+2,\ldots, n \ . \end{cases}$$

<u>Proof</u>:

Let $x^*(\lambda)$ be the solution to Eqs. 11 and 12 for $\lambda = \bar{\lambda}$.

Since $\bar{x}(\bar{\lambda})$ is optimal, by Lemma 3

$$\bar{\lambda} = f_i'(\bar{x}_i(\bar{\lambda})) \ , \quad i=1,2,\ldots, n \ .$$

Hence, by the concavity of $f_i(\cdot)$,

$$\Delta f_i([\bar{x}_i(\bar{\lambda})] + 1) \leq \bar{\lambda} \leq \Delta f_i([\bar{x}_i(\bar{\lambda})] - 1) \ .$$

Thus, by Lemma 6 $x_i^*(\bar{\lambda}) = [\bar{x}_i(\bar{\lambda})]$ or $[\bar{x}_i(\bar{\lambda})] + 1$.

Now if $\sum_{i=1}^{n} x_i^*(\bar{\lambda}) = m$, then $x_i^*(\lambda)$ is the optimal integer solution. In general this equality will not hold. For specificity suppose that $\sum_{i=1}^{n} x_i^*(\bar{\lambda}) > m$. The remaining case is treated analogously and is omitted.

By Theorem 8 and Corollary 9, increasing $\bar{\lambda}$ to some λ^* gives an optimal integer solution $x^*(\lambda^*)$. By Lemma 7, $x^*(\lambda)$ is decreasing in λ . Thus,

$$x_i^*(\lambda^*) \leq x_i^*(\bar{\lambda}) \ , \quad i=1,2,\ldots, n \ .$$

The proof is completed by contradiction. Suppose that

$$x_i^*(\lambda) = [\bar{x}_i(\bar{\lambda})] - \ell \geq 0$$

for some i where ℓ is a positive integer. Then there

must also exist at least one index j for which

$$x_j^*(\lambda^*) = [\bar{x}_j(\bar{\lambda})] + 1 \quad ,$$

for if not, then

$$\sum_{i=1}^{n} x_j^*(\lambda^*) < \sum_{i=1}^{n} [\bar{x}_j(\bar{\lambda})] \leq \sum_{i=1}^{n} \bar{x}_j(\bar{\lambda}) = m$$

which is nonoptimal for the integer problem. From the conditions for integer optimality (Lemma 6),

$$\Delta f_i([\bar{x}_i(\bar{\lambda})] - \ell - 1) > \lambda^* \geq \Delta f_i([\bar{x}_i(\bar{\lambda})] - \ell) \quad , \tag{18}$$

and

$$\Delta f_j([\bar{x}_j(\bar{\lambda})]) > \lambda^* \geq \Delta f_j([\bar{x}_j(\bar{\lambda})] + 1) \ . \tag{19}$$

It is now shown that

$$\Delta f_i([\bar{x}_i(\bar{\lambda})] - 1) \geq \Delta f_j([\bar{x}_j(\bar{\lambda})]) \tag{20}$$

for all i and j which will contradict Eq. 18 and Eq. 19, since then

$$\lambda^* \geq \Delta f_i([\bar{x}_i(\bar{\lambda})] - \ell) \geq \Delta f_i([\bar{x}_i(\bar{\lambda})] - 1)$$
$$> \Delta f_j([\bar{x}_j(\bar{\lambda})] > \lambda^* \ .$$

By the fundamental theorem of calculus

$$\int_{-1}^{0} f_i'(x_i + t)dt = f_i(x_i) - f_i(x_i - 1) \ .$$

By hypothesis, then

$$\int_{-1}^{0} f_i'(\bar{x}_i(\bar{\lambda}) + t)dt = \int_{-1}^{0} f_j'(\bar{x}_j(\bar{\lambda}) + t)dt$$

or

$$f_i(\bar{x}_i(\bar{\lambda})) - f_i(\bar{x}_i(\bar{\lambda}) - 1) = f_j(\bar{x}_j(\bar{\lambda})) - f_j(\bar{x}_j(\bar{\lambda}) - 1) \ .$$

By the concavity of $f_i(\cdot)$ and $f_j(\cdot)$,

$$\begin{aligned}
\Delta f_i([\bar{x}_i(\bar{\lambda})] - 1) &\geq f_i(\bar{x}_i(\bar{\lambda})) - f_i(\bar{x}_i(\bar{\lambda}) - 1) \\
&= f_j(\bar{x}_j(\bar{\lambda})) - f_j(\bar{x}_j(\bar{\lambda}) - 1) \\
&\geq \Delta f_j([x_j(\lambda)]) \quad ,
\end{aligned}$$

establishing Eq. 19.

To complete the proof it suffices, by appeal to Theorem 10, to show that $r_i \geq r_j$ implies that

$$\Delta f_i([\bar{x}_i(\bar{\lambda})]) \geq \Delta f_j([\bar{x}_j(\bar{\lambda})]) \quad .$$

If $r_i \geq r_j$, then the concavity of $f_j(\cdot)$ implies that

$$f_j'(\bar{x}_j(\bar{\lambda}) - r_j + t) \leq f_j'(\bar{x}_j(\bar{\lambda}) - r_i + t)$$

for all $t \geq r_i - \bar{x}_j(\bar{\lambda})$. Hence,

$$\begin{aligned}
&\Delta f_i([\bar{x}_i(\bar{\lambda})]) - \Delta f_j([\bar{x}_j(\bar{\lambda})]) \\
&\quad = \int_0^1 \left[f_i'([\bar{x}_i(\bar{\lambda})] + t) - f_j'([\bar{x}_j(\bar{\lambda})] + t) \right] dt \\
&\quad = \int_0^1 \left[f_i'(\bar{x}_i(\bar{\lambda}) - r_i + t) - f_j'(\bar{x}_j(\bar{\lambda}) - r_j + t) \right] dt
\end{aligned}$$

$$\geq \int_0^1 \left[f_i'(\bar{x}_i(\bar{\lambda}) - r_i + t) - f_j'(\bar{x}_j(\bar{\lambda}) - r_i + t) \right] dt$$

$$= 0 \quad . \qquad \text{Q.E.D.}$$

<u>Theorem 12</u>: There exist constants (α_i, β_i), when $x > 0$, such that for some j and all i

$$f_i(x + \alpha_i) + \beta_i = f_j(x)$$

if and only if

$$f_i'(x_i(\lambda) + \delta) = f_j'(x_j(\lambda) + \delta)$$

for all $\delta \geq 0$ where $x_i(\lambda) > 0$ and $x_j(\lambda) > 0$.

<u>Proof</u>:

By the definition of $x_i(\lambda)$,

$$f_i'(x_i(\lambda)) = f_j'(x_j(\lambda)) = \lambda$$

for $x_i(\lambda) > 0$ and $x_j(\lambda) > 0$. Thus, $f_i'(x_i(\lambda) + \delta) = f_j'(x_j(\lambda) + \delta)$ for all $\delta > 0$ is equivalent to

$$f_i(x_i(\lambda) + \delta) - f_i(x_i(\lambda)) = f_j(x_j(\lambda) + \delta) - f_j(x_j(\lambda))$$

for all $\delta \geq 0$. Letting $\alpha_i = x_i(\lambda) - x_j(\lambda)$ and $\beta_i = f_j(x_j(\lambda)) - f_i(x_i(\lambda))$, we have

$$f_i(x_j(\lambda) + \delta + \alpha_i) + \beta_i = f_j(x_j(\lambda) + \delta)$$

which completes the proof. Q.E.D.

V

EXAMPLES

In this section examples are given of the types of problems amenable to the analysis presented in the first four sections. Particular attention is paid to a problem which is commonly encountered in military systems analysis. The section concludes with a listing of some objective functions which meet the condition of Theorem 11 (and, consequently, Theorem 12).

We consider first the problem of finding a nonnegative integer allocation x so as to

$$\text{maximize} \quad \sum_{i=1}^{n} v_i[1 - p_i^{x_i}] \tag{21}$$

$$\text{subject to} \quad \sum_{i=1}^{n} x_i \leq m \quad . \tag{22}$$

This problem is familiar to many military analysts, particularly those for whom analysis of strategic systems is of concern. A protagonist has an inventory of m identical weapons which are to be targeted on some subset of n available targets. For each target there is a probability of p_i, $0 < p_i < 1$, that it survives a shot from one weapon. It is assumed then that $1 - p_i^{x_i}$ is the probability that the target is destroyed if x_i weapons are targeted on it. The parameter v_i is a measure of the value of target i. The objective is to maximize the expected value of the targets destroyed by m weapons. A discussion of such problems can be found in a recent paper by W. J. Schultis.[1]

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1. W. J. Schultis, A Manual Model for Strategic Conflict Analysis, IDA Research Paper P-493 (July 1969), pp. 15-20.

A model drawn from reliability theory also can be represented by Eqs. 21 and 22. Suppose there are n production lines. If line i operates without fail for a specified length of time, say T, then a return of v_i is realized. A line fails before time T if some subsystem of that line fails before time T. Suppose that $p_i(T)$ is the probability that the subsystem in production line i survives until time T. If parallel redundancy is allowed then $1 - [p_i(T)]^{x_i}$ is the probability that production line i survives until time T when x_i parallel subsystems are used. Again the objective is to maximize the expected return from the use of m subsystems.

Assuming that $v_i \geq 0$ and $0 < p_i < 1$, for $i = 1,2,\ldots, n$, the objective function given in Eq. 21 is strictly increasing and strictly concave. Applying Lemma 3, it is found that for $\lambda > 0$,

$$x_i(\lambda) = \left[\frac{\ell n(-\lambda/v_i \ell n p_i)}{\ell n p_i}\right]^+ , \qquad (23)$$

where $x^+ = \max(0, x)$. Using the fact that $\sum_{i=1}^{n} x_i(\lambda) = m$, the variable λ can be eliminated to find the continuous solution to Eqs. 21 and 22. Lemus and David[2] use Eq. 23 in conjunction with some heuristic rounding procedures to obtain an integer solution to the problem. In a recent IDA paper C. L. Dym and R. E. Schwartz[3] provide a means of computing the optimal integer solution based on the optimal continuous solution for the special case where $p_i = p$ for $i = 1, 2,\ldots, n$. In the terminology of Section IV they show that

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2. F. Lemus and K. H. David, "An Optimum Allocation of Different Weapons to a Target Complex," Operations Research, II (1963), pp. 787-794.

3. C. L. Dym and R. E. Schwartz, Optimum Pre-Attack Targeting, IDA Research Paper P-546 (October 1969).

the optimal integer solution is attainable from the optimal continuous solution by a rounding procedure based on the fractional part of the continuous solution.

To demonstrate the use of Theorems 10, 11, and 12 it is now shown that the Dym/Schwartz result can be obtained from the more general theory of Section IV.

Now $f_i(x_i) = v_i\,[1 - p_i^{x_i}]$ and so

$$f_i'(x_i) = -\,v_i(\ell n p_i)p_i^{x_i} \quad .$$

Then by using Eq. 23,

$$f_i'(x_i(\lambda) + \epsilon) = -\,v_i(\ell n p_i)p_i^{\ell n(-\lambda/v_i \ell n p_i)/\ell n p_i + \epsilon}$$

for $x_i(\lambda) > 0$. Simplifying,

$$f_i'(x_i(\lambda) + \epsilon) = -v_i(\ell n p_i)\left[\frac{-\lambda}{v_i \ell n p_i}\right] \cdot p_i^{\epsilon}$$
$$= \lambda p_i^{\epsilon} \quad .$$

Thus, it is seen that if $p_i \equiv p$, $f_i'(x_i(\lambda) + \epsilon) = f_j'(x_j(\lambda) + \epsilon)$. Hence, the hypotheses of Theorem 11 hold, and the optimal integer solution is readily found from the continuous solution by comparing the fractional parts of the variables.

In contrast to developing a rounding procedure, the optimal integer solution can be found directly by using the results given in Section III. Let

$$h_i(x) = \Delta f_i(x) \quad , \qquad i = 1,2,\ldots, n \quad .$$

Then

$$h_i(x) = v_i(1 - p_i)p_i^{x} \quad .$$

An algorithm which yields an optimal integer solution follows:

Step 1. Compute $h_i(x)$ and order these values in a decreasing sequence, say $k_1 \geq k_2 \geq \dots$. Take $\lambda = \max_{i,x} \{h_i(x)\} = k_1$. Set $r = 1$. Set $i = 1$.

Step 2. Replace i by $i + 1$. Replace r by $r + 1$. If $r < m$, then repeat Step 2. If $r = m$ go to Step 3.

Step 3. Recover the optimal allocation by letting x_i be the largest value of x for which $h_i(x)$ is included in the set $\{k_1, k_2, \dots, k_m\}$.

The proof of the algorithm is based on Corollary 9 and is not given here.

Examples of three other functional forms which satisfy the conditions of Theorem 11 are now given. First consider

$$f_i(x) = a_i(1 - b_i c^x)$$

where $a_i \geq 0$, $b_i \geq 0$, and $0 < c < 1$. Note that this function encompasses as a special case the one considered previously as well as the function $a_i(1 - e^{-cx})$. By considering derivatives, $f_i(x)$ is easily shown to be nondecreasing and concave. In fact

$$f_i'(x) = - a_i b_i (\ln c) c^x \quad ,$$

and so

$$x_i(\lambda) = \left[\frac{\ln[- \frac{\lambda}{a_i b_i \ln c}]}{\ln c} \right]^+$$

and $f_i'(x_i(\lambda) + \epsilon) = \lambda\, c^\epsilon$ which is independent of i.

As another example consider

$$f_i(x) = a \ln(b_i - c_i d^x) \quad ,$$

where $a \geq 0$, $b_i > c_i \geq 0$, and $0 < d < 1$. Again $f_i(x)$ can be shown to be nondecreasing and concave. Furthermore,

$$x_i(\lambda) = \left[\frac{\ln[\frac{\lambda b_i}{\lambda c_i - ac_i \ln d}]}{\ln d} \right]^+ ,$$

so that

$$f_i'(x_i(\lambda) + \epsilon) = - \frac{\lambda a(\ln d)d^\epsilon}{\lambda - a(\ln d) - \lambda d^\epsilon}$$

which is independent of i.

Finally suppose that

$$f_i(x) = a \ln(b_i + c_i x) ,$$

where $a \geq 0$, $b_i \geq 0$, and $c_i \geq 0$. Then $f_i(x)$ is nondecreasing and concave and

$$f_i'(x) = \frac{ac_i}{b_i + c_i x} .$$

It is easily shown that

$$x_i(\lambda) = \left[\frac{ac_i - \lambda b_i}{\lambda c_i} \right]^+ ,$$

and that

$$f_i'(x_i(\lambda) + \epsilon) = \frac{\lambda a}{a + \lambda \epsilon} ,$$

which is independent of i.